Geometric Basics

Introduction to Computer Graphics
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Graphics Pipeline

Hardware

Modelling → Transform → Visibility

Illumination + Shading

Perception, Interaction

Color

Texture/Realism
Reading

- Chapter 4 of Angel
- Chapter 7 of Foley, van Dam, …
- Chapter 2 of Marschner & Shirley
Schedule

- Geometry basics
- Affine transformations
- Use of homogeneous coordinates
- Concatenation of transformations
- 3D transformations
- Transformation of coordinate systems
- Transform the transforms
- Transformations in OpenGL

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Why do we need transformations?

- Need to transform objects from object coordinate system (OCS) to world CS
- Need to transform objects from world CS to the coordinate system of the camera (ViewCS)
- Need to transform objects from VCS into an WebGL window/viewport (DisplayCS)
- Object may move (translate or rotate) or deform (scale, shear, or general non-rigid deformation)
Geometry basics

• Scalar, point, and vector
• Vector space and affine space
• Basic point and vector operations
• Sided-ness test
• Lines, planes, and triangles
• Linear independence
• Coordinate systems and frames
Scalar, point, and vector

- **Point**: a location in space
  - Specified by a k-tuple for k-d points
  - Always given with respect to some coordinate system
- **Scalar**: a quantity, e.g., edge length
- **Vector**: a directed line segment between points
- **Spaces**: vector space, affine space, Euclidean space, etc.

\[ P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \]
Vector space

• A set of vectors with scalar multiplications and vector additions
  – Scalar-vector multiplication: $u = \alpha v$
  – Vector-vector addition: $w = u + v$

• Expressions such as $v = u + 2w - 3r$ make sense in a vector space

• But vectors lack position
  – Inadequate for representing geometry – we need positions, which are given by points
Affine space

• A vector space + points = affine space

• Operations
  – Vector-vector addition
  – Scalar-vector multiplication
  – Point-vector addition
  – Affine sum of points and convex sums

• A vector space + distance/norm = Euclidean space
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Basic point and vector operations

- point – point = vector
- point + vector = point
- vector operations:
  - scalar * vector = vector
  - vector + vector = vector
  - vector ⋅ vector = scalar, the dot product
  - vector × vector = vector, the cross product

Right-hand rule
More on dot product

- \( \mathbf{u} \cdot \mathbf{v} = u_x \cdot v_x + u_y \cdot v_y + u_z \cdot v_z \)
- \( \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \) is always non-negative
- \( \mathbf{u} \cdot \mathbf{v} \) is commutative and distributive over additions
- \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta \)
- Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \)
- If \( \mathbf{v} \) is normalized, i.e., \( |\mathbf{v}| = 1 \), then \( \mathbf{u} \cdot \mathbf{v} \) gives the projection of \( \mathbf{u} \) in the direction of \( \mathbf{v} \)
More on cross product

• Cross product $u \times v$ is a vector perpendicular to $u$ and $v$ – frequently used to compute the **normal to a plane**

• Direction of the cross product is determined by the **right hand rule**

• $u \times v = -v \times u$

• $|u \times v| = |u| \cdot |v| \cdot |\sin \theta| = \text{area of the parallelogram}$

• How to compute? – use determinant

\[
\begin{vmatrix}
  i & j & k \\
  u_x & u_y & u_z \\
  v_x & v_y & v_z \\
\end{vmatrix}
\]
Affine and convex sums

• Addition of two arbitrary points is not defined in an affine space

• But consider two points P and Q with \( Q = P + \alpha v \), we can always find a point R such that \( v = R - P \) so now we have

\[ Q = P + \alpha(R - P) \text{ or } Q = \alpha R + (1 - \alpha)P \]

• Thus, **affine sum** (combination) of points can be defined

\[ t_1P_1 + t_2P_2 + \ldots + t_nP_n, \quad t_1 + t_2 + \ldots + t_n = 1 \]

• **Convex sum** (combination) of points

\[ t_1P_1 + \ldots + t_nP_n, \quad \text{where} \]

\[ t_1 + \ldots + t_n = 1 \text{ and } t_i \geq 0 \text{ for all } i \]
Convex hull

- **Convex hull** of a set of points: set of convex combination of these points

- Alternatively, the convex hull is the smallest convex object containing the set of points

- Formed by “shrink wrapping” points

- Useful in, e.g., fast collision detection
Sided-ness test

• On which side does a point $V$ lie with respect to a line, specified by a vector $u$?

• Solution 1: use an implicit line or plane equation; plug in the point coordinates and check the sign

• Solution 2: in 2D, let the $z$ coordinate be zero, compute a cross product and check the sign of the $z$ component

• Solution 3: find a vector $u'$ perpendicular to $u$; check sign of $u' \cdot v$
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Line representations

• Consider all points of the form

\[ P(\alpha) = P_0 + \alpha d \]

this is the set of all points lying on a line that passes through \(P_0\) in the direction of the vector \(d\)

• Known as the parametric form of the line
  – Given two points \(R\) and \(Q\) on the line, we have
  \[
  x(\alpha) = \alpha R_x + (1 - \alpha) Q_x \\
y(\alpha) = \alpha R_y + (1 - \alpha) Q_y
  \]

• Other representations
  – Explicit: \(y = mx + h\)
  – Implicit: \(ax + by + c = 0\)
Ray and line segment

- If $\alpha \geq 0$, then $P(\alpha)$ is the ray leaving $P_0$ in the direction of the vector $d$

- For the two-point representation

$$ x(\alpha) = \alpha R_x + (1 - \alpha) Q_x $$

$$ y(\alpha) = \alpha R_y + (1 - \alpha) Q_y $$

if $1 \geq \alpha \geq 0$, then we get all the points on the line segment joining $R$ and $Q$
Parametric plane representation

- A plane can be defined by a point and two vectors or by three points

\[ P(a, b) = R + au + bv \]

\[ P(a, b) = R + a(Q - R) + b(P - R) \]
Implicit plane equation

\[ F(x, y, z) = ax + by + cz + d = 0 \]

- Typically, \((a, b, c)\) is the unit normal of the plane
- If \(F(x_0, y_0, z_0) < 0\), point \((x_0, y_0, z_0)\) is below the plane (w. r. t. the normal)
- If \(F(x_0, y_0, z_0) > 0\), point \((x_0, y_0, z_0)\) is above the plane
- In general, the distance from a point \((x_0, y_0, z_0)\) to the plane is given by
  \[
  \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}
  \]
Triangle and barycentric coordinates

T is a convex sum of P, Q, and R. The weights are called the \textbf{barycentric coordinates} of T.
Coordinate system (CS) and frame

• n linearly independent vectors of an n-D vector space define a coordinate system (CS), e.g., Cartesian CS
  – The vectors are called the basis vectors
  – Any vector in the space can be written as a linear sum of the basis vectors in a unique way

• An origin O, the reference point, along with a set of basis vectors form a frame
  – Any point P = O + linear sum of basis vectors
  – Coefficients of the sum: coordinates of point P
From one CS to another

- Express point given in one frame/CS in another CS
- In 2D, need to solve system of two equations

Assume the two CS have the same origin

\[
P = (1, 2)^T \quad \text{Basis: } (1, 1)^T \text{ and } (0, 1)^T
\]

\[
P' = (x, y)^T \quad \text{New set of basis: } (-1,1)^T \text{ and } (1, 0)^T
\]

Need to find \(x\) and \(y\) such that

\[
1*(1,1)^T + 2*(0,1)^T = x*(-1,1)^T + y*(1, 0)^T
\]

Alternatively

\[
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
2
\end{bmatrix}
=
\begin{bmatrix}
-1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]
Change of CS

• With the 2D basis vectors \( u \) (\( u' \)) and \( v \) (\( v' \)), find a 2 \times 2 \) matrix \( M \) such that

\[
[u \ v] M = [u' \ v'], \text{ thus } M = [u \ v]^{-1}[u' \ v']
\]

Then \( P(u', v') = M^{-1}P(u, v) \) for all points \( P \)

• The matrix transforms points from one CS, with basis \( u \) and \( v \), to another CS, with basis \( u' \) and \( v' \)

• \( M \) is also called the “change of basis” matrix